The solvable three-dimensional rank-two separable potential model: partial-wave scattering

2008 J. Phys. A: Math. Theor. 41 275306

(http://iopscience.iop.org/1751-8121/41/27/275306)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 275306 (11pp) doi:10.1088/1751-8113/41/27/275306

The solvable three-dimensional rank-two separable potential model: partial-wave scattering

A Maghari and M Dargahi

Department of Physical Chemistry, School of Chemistry, College of Science, University of Tehran, Tehran, Iran

E-mail: maghari@khayam.ut.ac.ir

Received 7 January 2008, in final form 14 May 2008
Published 16 June 2008
Online at stacks.iop.org/JPhysA/41/275306

Abstract

In this paper, we study the ℓth partial-wave scattering for particles subjected to a non-local rank-two separable potential. Analytic expressions for the scattering amplitude, bound and resonance states, phase shifts and time delays are obtained and numerical illustrations are provided.

PACS numbers: 03.65.Nk, 03.65.Fd, 03.65.Db, 03.65.Ge

1. Introduction

Nonlocal two-body separable interactions have often been used in many-body problems because of the fact that the two-body Lippmann–Schwinger equation is exactly solvable [1–5], and leads to closed expressions for a large class of such interactions. These potentials have also been used with Faddeev equations for the three-body problem [6–8].

The theory of separable potentials was first proposed for the $^3S_1$ nucleon–nucleon interaction by Yamaguchi [9, 10]. The behavior of a certain kind of nonlocal potential has been studied by Mitra *et al* [11–13] in the complex angular momentum plane. Tabakin [14, 15] used a set of separable potentials with small off-energy-shell T-matrix elements, matching the s, p and d-wave nucleon–nucleon phase parameters without generating the usual strong short-range correlations. Mongan [16, 17] also used a set of rank-two separable potential of the same general form as Tabakin's, which fit the Arndt–MacGregor nucleon–nucleon phase parameters [18] for partial waves through $\ell = 4$. Ghirardi and Rimini [19] have examined general properties of separable potentials. Moreover, such potentials have been studied as models for a variety of physical problems [20–30].

In a recent work, the s-wave scattering properties of a system with zero angular momentum (s-wave) via a rank-two separable potential were calculated [31] and used for obtaining the quantum statistical mechanical properties of fluids [32]. In this paper, we have been motivated to extend our previous work for a system with arbitrary angular momentum via a rank-two separable potential. In the present work, the formalism is generalized to obtain the explicit analytical results for the ℓth partial-wave scattering wavefunction and its properties. It should be mentioned that most contributions appearing in the literature are concerned with the s-wave case.

2. Separable potential

A separable potential model can generally be written as

$$\hat{V} = \sum_{i=1}^{n} \sum_{\ell} (2\ell + 1) v_i |\chi_i; \ell\rangle\langle\ell; \chi_i| \tag{2.1}$$

where n is the rank of the potential operator $\hat{V}$, v_i is the attractive (or repulsive) coupling strength and $|\chi_i; \ell\rangle$ is the state of the system with angular momentum quantum number ℓ, which is a real number in the unitary case. The momentum representation of such a potential is

$$\hat{V}(\boldsymbol{p}, \boldsymbol{p}') = \langle \boldsymbol{p}|\hat{V}|\boldsymbol{p}'\rangle = \sum_{i=1}^{n} \sum_{\ell} (2\ell + 1) v_i \chi_i^{(\ell)^*}(\boldsymbol{p}')\chi_i^{(\ell)}(\boldsymbol{p}) \tag{2.2}$$

where $\chi_i^{(\ell)}(\boldsymbol{p}) \equiv \langle \boldsymbol{p}|\chi_i; \ell\rangle$. A possible generalized separable potential for the ℓth partial wave can be written as a Yamaguchi type

$$\chi_i^{(\ell)}(\boldsymbol{p}) = \frac{1}{\pi^{3/4}} \left[\frac{2^{2\ell}\ell!(2\ell + 1)a_i^{2\ell+1}}{\Gamma(\ell + 1/2)}\right]^{1/2} \frac{p^{\ell}}{\left(a_i^2 + p^2\right)^{\ell+1}} \mathrm{P}_{\ell}(\cos\theta_p) \tag{2.3}$$

where $\mathrm{P}_{\ell}(\cos\theta_p)$ is the Legendre function in which θ_p is the polar angle of vector $\boldsymbol{p}$, $\Gamma(n)$ is the Gamma function and a_i represents the inverse range of the potential.

The state-dependent form factor of our separable potential model $\chi_i^{(\ell)}(\boldsymbol{r})$ can be calculated from the Fourier transform of equation (2.3) as

$$\chi_i^{(\ell)}(\boldsymbol{r}) \equiv \langle \boldsymbol{r}|\chi_i^{(\ell)}\rangle = \frac{2\pi^{1/4}}{h^{3/2}} \left[\frac{2^{2\ell}\ell!(2\ell + 1)a_i^{2\ell+1}}{\Gamma(\ell + 1/2)}\right]^{1/2} \int_C \mathrm{d}p \frac{p^{\ell+2}}{\left(a_i^2 + p^2\right)^{\ell+1}} \mathrm{d}p \times \int_{-1}^{1} \mathrm{d}(\cos\theta_p)\, \mathrm{P}_{\ell}(\cos\theta_p)\, \mathrm{e}^{\mathrm{i}pr\cos\theta_{pr}/\hbar} \tag{2.4}$$

where C is a contour in the complex p-plane which goes from $-\infty$ to ∞ passing above the poles and θ_{pr} is the angle between $\boldsymbol{p}$ and $\boldsymbol{r}$. Using the Jacobi–Anger expansion [33]

$$\mathrm{e}^{\mathrm{i}pr\cos\theta/\hbar} = \sum_{n=-\infty}^{\infty} \mathrm{i}^n J_n(pr/\hbar)\, \mathrm{e}^{\mathrm{i}n\theta} = \mathrm{J}_0(pr/\hbar) + 2\sum_{n=1}^{\infty} \mathrm{i}^n \mathrm{J}_n(pr/\hbar) \cos(n\theta) \tag{2.5}$$

the form factor $\chi_i^{(\ell)}(\boldsymbol{r})$ can then be written as

$$\chi_i^{(\ell)}(\boldsymbol{r}) = \frac{4\pi^{1/4}}{h^{3/2}} \left[\frac{2^{2\ell}\ell!(2\ell + 1)a_i^{2\ell+1}}{\Gamma(\ell + 1/2)}\right]^{1/2} \mathrm{P}_{\ell}(\cos\theta_r) \sum_{n=0} \mathrm{i}^n g^{(\ell)}(n) \int_C \mathrm{d}p \frac{p^{\ell+2}}{\left(a_i^2 + p^2\right)^{\ell+1}} \mathrm{J}_n(pr/\hbar) \tag{2.6}$$

where $\mathrm{J}_n(x)$ is a Bessel function of the first kind of integral order n and $g^{(\ell)}(n)$ is a function of n, which can be expressed as:

$$g^{(\ell)}(n) = \begin{cases} \displaystyle\prod_{k=0}^{(\ell-2)/2} \prod_{k'=0}^{\ell/2} \frac{[n^2 - (2k)^2]}{[n^2 - (2k' + 1)^2]} (-\cos n\pi - 1) & \ell = \text{even} \\ \displaystyle\prod_{k=0}^{(\ell-3)/2} \prod_{k'=0}^{(\ell+1)/2} \frac{[n^2 - (2k + 1)^2]}{[n^2 - (2k')^2]} (-\cos n\pi - 1) & \ell = \text{odd} \end{cases} \tag{2.7}$$

for $n \neq 0$ and $g^{(\ell)}(n) = \delta_{0\ell}/2$ for $n = 0$, where δ_{ij} is Kronecker delta. From the numerical evaluation of equation (2.7), it can be shown that a rapid absolute convergence of alternating series, appearing in equation (2.6), occurs. Moreover, it is convenient to introduce an integral form of the Bessel function

$$\mathrm{J}_n(x) = \frac{1}{2\pi \mathrm{i}}\left(\frac{x}{2}\right)^n \int t^{-n-1}\,\mathrm{e}^{(t-x^2/4t)}\,\mathrm{d}t. \tag{2.8}$$

The advantage in using the Jacobi–Anger expansion (equation (2.5)) in terms of the Bessel function $\mathrm{J}_n(x)$, rather than spherical Bessel function is that the integral in equation (2.6) may be solved analytically by substituting the integral form of $\mathrm{J}_n(x)$ (equation (2.8)) into equation (2.6). Therefore, the form factor leads to

$$\chi_i^{(\ell)}(\boldsymbol{r}) = \frac{2\pi^{-3/4}}{h^{3/2}}\left[\frac{2^{2\ell}\ell!(2\ell+1)a_i^{2\ell+1}}{\Gamma(\ell+1/2)}\right]^{1/2} \mathrm{P}_\ell(\cos\theta_r)$$
$$\times \sum_{n=0} \mathrm{i}^{n-1} g^{(\ell)}(n)\left(\frac{r}{2\hbar}\right)^n \int \mathrm{e}^t \frac{\mathrm{d}t}{t^{n+1}} \int_C \mathrm{d}p \left[\frac{p}{a_i^2+p^2}\right]^{\ell+1} p^{n+1} \exp\left(\frac{-r^2p^2}{4\hbar^2 t}\right) \tag{2.9}$$

The expression of $\chi_i^{(\ell)}(\boldsymbol{r})$ can be obtained using the contour integration by closing in the upper half of the complex plane. It can be shown that

$$\int_C \mathrm{d}p \left[\frac{p}{a_i^2+p^2}\right]^{\ell+1} p^{n+1} \exp\left(\frac{-r^2p^2}{4\hbar^2 t}\right) = \pi\mathrm{i}\left[\exp\left(\frac{-r^2a_i^2}{4\hbar^2 t}\right)\sum_{j=0}^{\ell} \xi_j^{(\ell,n)} G_{\ell-j}(r,t)\right] \tag{2.10}$$

where

$$\xi_j^{(\ell,n)} = \frac{(\ell+1)(\ell+2)}{2}\,\frac{(-1)^j(\mathrm{i}a_i)^{n+\ell+2-j}}{(2\mathrm{i}a_i)^{\ell+1+j}}\binom{n+\ell+2}{j} \tag{2.11}$$

$$G_{\ell-j}(r,t) = \sum_{k=0}^{\ell-j} \frac{(-1)^k}{k!}\,\frac{(2\mathrm{i}a_i)^{\ell-j-2k}}{(\ell-j-2k)!}\left(\frac{r^2}{4\hbar^2 t}\right)^{\ell-j-k} \tag{2.12}$$

in which $\binom{k}{j}$ is the binomial coefficient. Substituting equations (2.10)–(2.12) into (2.9), the form factor $\chi_i^{(\ell)}(\boldsymbol{r})$ can then be obtained from the following analytical expression:

$$\chi_i^{(\ell)}(\boldsymbol{r}) = (a_i/\hbar)^{3/2}(\ell+1)(\ell+2)\left[\frac{2^{2\ell+3}\ell!(2\ell+1)}{\sqrt{\pi}\,\Gamma(\ell+1/2)}\right]^{1/2} P_\ell(\cos\theta_r)$$
$$\times \sum_{n=0}^{\ell l}\sum_{j=0}^{\ell-j}\sum_{k=0} \frac{(-1)^n i^{\ell l+j}}{2^{k+j}}\,\frac{1}{k!}\,\frac{1}{(\lambda-k)!}\binom{n+\ell+2}{j} g^{(\ell)}(n)\left(\frac{ra_i}{\hbar}\right)^{\lambda} \mathrm{J}_\lambda(ra_i/\hbar) \tag{2.13}$$

where $\lambda \equiv \ell - j - k$.

3. The scattering states

The Lippmann–Schwinger equation for the partial-wave scattering state associated with the incident plane wave $|\varphi\rangle$ is as follows [34]:

$$\left|\psi_z^{(\ell)}\right\rangle = |\varphi\rangle + \frac{1}{z-\hat{H}_0}\hat{V}^{(\ell)}\left|\psi_z^{(\ell)}\right\rangle \tag{3.1}$$

where $\hat{H}_0 = \hat{p}^2/2m$ is free motion Hamiltonian and $z \equiv E + \mathrm{i}\varepsilon$ is the complex-energy parameter.

The exact solution of the Lippmann–Schwinger equation (equation (3.1)) via the separable potential plays an important role in quantum scattering theory, since it contains the information for studying the many-body systems. In this work, the partial-wave scattering properties of a system have been calculated from the analytical solution of the Lippmann–Schwinger equation via a nonlocal rank-two separable potential. By using equation (3.1), the components of the scattering wavefunction in the $\chi_1^{(\ell)}$ and $\chi_2^{(\ell)}$ channels can be obtained from the following system of equations:

$$\begin{cases} \left(v_1 Q_{11}^{(\ell)} - 1\right) B_1^{(\ell)} + v_2 Q_{12}^{(\ell)} B_2^{(l)} + C_1^{(\ell)} = 0 \\ v_1 Q_{21}^{(\ell)} B_1^{(\ell)} + \left(v_2 Q_{22}^{(\ell)} - 1\right) B_2^{(\ell)} + C_2^{(\ell)} = 0 \end{cases} \tag{3.2}$$

where $C_i^{(\ell)} (i = 1, 2)$ is the inner product $\langle \chi_i; \ell | \varphi \rangle$ and the matrix elements of the free motion resolvent $Q_{ij}^{(\ell)}$ are defined for any complex z as follows:

$$Q_{ij}^{(\ell)}(z) \equiv \langle \chi_i; \ell | \frac{1}{z - \hat{H}_0} | \chi_j; \ell' \rangle \delta_{\ell\ell'} \qquad (i, j = 1, 2). \tag{3.3}$$

The solution of equation (3.2) leads to

$$\begin{pmatrix} B_1^{(\ell)} \\ B_2^{(\ell)} \end{pmatrix} = \frac{1}{D} \begin{pmatrix} 1 - v_2 Q_{22}^{(\ell)} & v_2 Q_{12}^{(\ell)} \\ v_1 Q_{21}^{(\ell)} & 1 - v_1 Q_{11}^{(\ell)} \end{pmatrix} \begin{pmatrix} C_1^{(\ell)} \\ C_2^{(\ell)} \end{pmatrix} \tag{3.4}$$

where the determinant D is defined as

$$D = \left(1 - v_1 Q_{11}^{(\ell)}\right)\left(1 - v_2 Q_{22}^{(\ell)}\right) - v_1 v_2 Q_{12}^{(\ell)^2}. \tag{3.5}$$

The matrix elements $Q_{ij}^{(\ell)}(z)$ can be obtained in the upper half q-plane as

$$\begin{aligned} Q_{ij}^{(\ell)}(q) &= 2m \int_C \frac{\chi_i^{(\ell)^*}(\boldsymbol{p}) \chi_j^{(\ell)}(\boldsymbol{p})}{(q^2 - p^2)} \, \mathrm{d}\boldsymbol{p} \\ &= 4m\pi^{-1/2} (a_i a_j)^{(2\ell+1)/2} \left[\frac{2^{2\ell} \ell!}{\Gamma(\ell + 1/2)} \right] \int_C \left[\frac{p^2}{\left(a_i^2 + p^2\right)\left(a_j^2 + p^2\right)} \right]^{\ell+1} \frac{\mathrm{d}p}{(q^2 - p^2)} \end{aligned} \tag{3.6}$$

where $q \equiv \sqrt{2mz}$. The expression of $Q_{ij}^{(\ell)}(z)$ can be evaluated using the contour integration by closing in the upper half of the complex plane. The result is

$$\tilde{Q}_{ij}^{(\ell)}(\tilde{q}) = \frac{2^{\ell+2} \ell! \pi^{1/2}}{\Gamma(\ell + 1/2)} \frac{(\tilde{a}_i \tilde{a}_j)^{\frac{2\ell+3}{2}}}{(\tilde{a}_i + \tilde{a}_j)^{2\ell+1}} \frac{\Lambda^{(\ell)}(\tilde{a}_i, \tilde{a}_j, \tilde{q})}{(\tilde{q} + i\tilde{a}_i)^{\ell+1} (\tilde{q} + i\tilde{a}_j)^{\ell+1}} \tag{3.7}$$

where $\tilde{Q}_{ij}^{(\ell)}(\tilde{q}) = (a_i a_j/m) Q_{ij}^{(\ell)}(\tilde{q})$ are the reduced matrix elements of $Q_{ij}^{(\ell)}(\tilde{q})$, in which $\tilde{q} \equiv q/a_{av}$ is a dimensionless momentum variable and $\tilde{a}_{i(j)} \equiv a_{i(j)}/a_{\mathrm{av}}$ is a dimensionless inverse range parameter, in which $a_{\mathrm{av}} \equiv \frac{a_1 + a_2}{2}$. Equation (3.7) is the key result that allows a complete analytical solution of the model potential. In appendix A, the analytic expressions of $\Lambda^{(\ell)}(\tilde{a}_i, \tilde{a}_j, \tilde{q})$ are given.

The momentum representation of the partial-wave scattering wavefunction $\psi_z^{(\ell)}(\tilde{\boldsymbol{p}}) \equiv \left\langle \tilde{\boldsymbol{p}} \,\middle|\, \psi_z^{(\ell)} \right\rangle$ can be evaluated from the model potential (2.2) and equation (3.1) as

$$\begin{aligned} \psi_z^{(\ell)}(\tilde{\boldsymbol{p}}) = \varphi(\tilde{\boldsymbol{p}}) &+ \frac{1}{D} \frac{2^{\ell+1}}{\pi^{3/4}} \left[\frac{\ell!(2\ell + 1)}{\Gamma(\ell + 1/2)} \right]^{1/2} \frac{\tilde{p}^\ell}{\tilde{q}^2 - \tilde{p}^2} P_\ell(\cos\theta_p) \\ &\times \sum_{i=1 (j \neq i)}^{2} \frac{\tilde{v}_i \tilde{a}_i^{(2\ell+5)/2}}{(\tilde{p}^2 + \tilde{a}_i^2)^{\ell+1}} \left[\left(1 - \tilde{v}_j \tilde{Q}_{ij}^{(\ell)}\right) \tilde{C}_i^{(\ell)} + (\tilde{a}_j/\tilde{a}_i) \tilde{v}_j \tilde{Q}_{ij}^{(\ell)} \tilde{C}_j^{(\ell)} \right] \end{aligned} \tag{3.8}$$

where $\tilde{v}_i \equiv mv_i/a_i^2$ is the dimensionless potential strength parameter and $\varphi(\tilde{\boldsymbol{p}})$ is the incoming wavefunction, which is usually the plane wave. Clearly, for $\ell = 0$ (s-wave), the scattering wavefunction $\psi_z^{(0)}(p)$ is independent of any direction of incident beam [31]. The corresponding coordinate representation of the partial-wave scattering wavefunction $\psi_z^{(\ell)}(\tilde{\boldsymbol{r}}) \equiv \langle \tilde{\boldsymbol{r}} \mid \psi_z^{(\ell)} \rangle$ can be obtained by a Fourier transformation

$$\psi_z^{(\ell)}(\boldsymbol{r}) = \frac{1}{(2\pi\hbar)^{3/2}} \int \psi_z^{(\ell)}(\boldsymbol{p})\, \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}/\hbar}\, \mathrm{d}\boldsymbol{p}. \tag{3.9}$$

Inserting equation (3.8) into equation (3.9), after performing integration over momentum analytically, the scattering wavefunction in coordinate representation becomes

$$\begin{aligned}\psi_z^{(\ell)}(\tilde{\boldsymbol{r}}) = \psi_0(\tilde{\boldsymbol{r}}) + \frac{2\sqrt{2}}{\pi^{1/4}} \frac{1}{D} \left[\frac{\ell!(2\ell+1)}{\Gamma(\ell+1/2)}\right]^{1/2} \sum_{\substack{i=1\\(i\neq j)}}^{2} \sum_{k=0}^{\ell} \sum_{n=0}^{[(\ell-k)/2]} 2^{\ell-n} \frac{(-1)^n}{n!} \frac{\mathrm{i}^{\ell-k-2n}}{(\ell-k-2n)!} \\ \times \frac{\xi_k^{(\ell+1)}}{\tilde{a}_i^{\ell-k-1/2}} \left[\left(1-\tilde{v}_j \tilde{Q}_{ij}^{(\ell)}\right)\tilde{C}_i^{(\ell)} + (\tilde{a}_j/\tilde{a}_i)\tilde{v}_j \tilde{Q}_{ij}^{(\ell)} \tilde{C}_j^{(\ell)}\right] \\ \times (\tilde{r}\tilde{a}_i)^{\ell-k-n} J_{2\ell-k-n}(\tilde{r}\tilde{a}_i) \mathrm{P}_\ell(\cos\theta)\end{aligned} \tag{3.10}$$

where $\tilde{C}_i^{(\ell)} \equiv a_{av}^{-3/2}\chi_i^{(\ell)}(\boldsymbol{r})$ is the reduced partial-wave form factor and $\tilde{\boldsymbol{r}} \equiv \frac{\bar{a}}{\hbar}\boldsymbol{r}$ is the reduced coordinate.

4. Transition matrix

We now turn to investigate the analytical properties of the transition matrix in the complex q-plane. The transition matrix explicitly shows the contributions from the bound states, resonances and distant singularities in the complex-energy plane. The ℓth partial-wave off-shell transition matrix elements in the momentum representation may be written as

$$\begin{aligned}\langle p|\hat{T}^{(\ell)}(z)|p'\rangle &= \langle p|\hat{V}^{(\ell)}[\hat{1} + (\hat{1} - \hat{Q}^{(\ell)}\hat{V}^{(\ell)})^{-1}\hat{Q}^{(\ell)}\hat{V}^{(\ell)}]|p'\rangle \\ &= \langle p|\hat{V}^{(\ell)}(\hat{1} - \hat{Q}^{(\ell)}\hat{V}^{(\ell)})^{-1}|p'\rangle.\end{aligned} \tag{4.1}$$

It is useful to introduce an arbitrary operator $\hat{K}$ and its reciprocal $\hat{K}^{-1}$ as

$$\hat{K} = \sum_{i=1}^{2} \alpha_i |\chi_i;\ell\rangle\langle\ell;\chi_i| \tag{4.2}$$

$$\hat{K}^{-1} = \sum_{i=1}^{2} \beta_i |\chi_i;\ell\rangle\langle\ell;\chi_i| \tag{4.3}$$

where α_i and β_i are parameters that must be satisfied by the following relations,

$$\sum_{i=1}^{2}\sum_{j=1}^{2} \gamma_{ij} \left[\frac{\tilde{a}_i\tilde{a}_j}{(\tilde{a}_i+\tilde{a}_j)(\tilde{a}_i+\tilde{a}_m)(\tilde{a}_j+\tilde{a}_n)}\right]^{2\ell+1} = -\frac{\pi}{2^{4\ell}(\ell!)^2} \frac{1}{(\tilde{a}_m+\tilde{a}_n)^{2\ell+1}} \tag{4.4}$$

where $\gamma_{ij} \equiv \alpha_i\beta_j$. Inserting $\hat{K}\hat{K}^{-1} = \hat{1}$ into equation (4.1) and using equations (4.2) and (4.3), the ℓth partial-wave off-shell transition matrix elements can be obtained as

$$\begin{aligned}\langle\tilde{p}|\hat{T}^{(\ell)}(z)|\tilde{p}'\rangle = T_{pp'}^{(\ell)}(z) &= \langle\tilde{p}|\hat{V}^{(\ell)}(\hat{1} - \hat{Q}^{(\ell)}\hat{V}^{(\ell)})^{-1}\hat{K}\hat{K}^{-1}|\tilde{p}'\rangle \\ &= \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} \gamma_{ij}\tilde{a}_k^2\tilde{v}_k\chi_k^{(\ell)}(\tilde{p})\chi_j^{(\ell)*}(\tilde{p}')\langle\chi_i^{(\ell)} \mid \chi_j^{(\ell)}\rangle(\hat{1} - \hat{\tilde{Q}}^{(\ell)}\hat{\tilde{V}}^{(\ell)})_{ki}^{-1}\end{aligned} \tag{4.5}$$

where

$$\langle \chi_i^{(\ell)} \mid \chi_j^{(\ell)} \rangle = \frac{2^{2\ell}\ell!}{\mathrm{i}\pi^{1/2}} \left(\frac{\sqrt{\tilde{a}_i \tilde{a}_j}}{\tilde{a}_i + \tilde{a}_j} \right)^{2\ell+1} \tag{4.6}$$

and the matrix elements $(\hat{1} - \hat{Q}^{(\ell)} \hat{V}^{(\ell)})^{-1}_{ij} = \langle \chi_i^{(\ell)} | (\hat{1} - \hat{Q}^{(\ell)} \hat{V}^{(\ell)})^{-1} | \chi_j^{(\ell)} \rangle$ can be obtained as

$$(\hat{1} - \hat{\tilde{Q}}^{(\ell)} \hat{\tilde{V}}^{(\ell)})^{-1}_{ki} = \frac{1}{D^{(\ell)}} \sum_{m=1}^{2} \langle \chi_m^{(\ell)} \mid \chi_i^{(\ell)} \rangle \left(\frac{\tilde{v}_m \tilde{a}_m}{\tilde{a}_k} \tilde{Q}_{km}^{(\ell)} - \delta_{km} \right) \qquad k \neq i \tag{4.7a}$$

$$(\hat{1} - \hat{\tilde{Q}}^{(\ell)} \hat{\tilde{V}}^{(\ell)})^{-1}_{11} = \frac{1}{D^{(\ell l)}} \sum_{m=1}^{2} \langle \chi_m^{(\ell)} \mid \chi_2^{(\ell)} \rangle \left(\delta_{m2} - \frac{\tilde{v}_m \tilde{a}_m}{\tilde{a}_2} \tilde{Q}_{2m}^{(\ell)} \right) \tag{4.7b}$$

$$(\hat{1} - \hat{\tilde{Q}}^{(\ell)} \hat{\tilde{V}}^{(\ell)})^{-1}_{22} = \frac{1}{D^{(\ell)}} \sum_{m=1}^{2} \langle \chi_m^{(\ell)} \mid \chi_1^{(\ell)} \rangle \left(\delta_{m1} - \frac{\tilde{v}_m \tilde{a}_m}{\tilde{a}_1} \tilde{Q}_{1m}^{(\ell)} \right) \tag{4.7c}$$

where $\tilde{Q}_{ij}^{(\ell)}$ are the reduced matrix elements of the free motion resolvent given in equation (3.7) and $D^{(\ell)} \equiv \det[1 - \hat{\tilde{Q}}^{(\ell)} \hat{\tilde{V}}^{(\ell)}]$ is the Fredholm determinant. The above analysis makes clear that the present method for dealing with scattering via the rank-two separable potentials allows an insight into the influence of various parts of the potential on the transition matrix.

5. Results and discussion

The method proposed in this paper applies to arbitrary angular momentum and rank-two separable potential. We shall illustrate the results of the preceding section for some arbitrary values of angular momentum via the rank-two separable potential, which is a combination of attractive and repulsive interactions with some values of inverse range parameter and coupling strength. In the present study, the values of the parameters for our potential model have been chosen for two cases: (i) $\tilde{v}_1 = 10$, $\tilde{v}_2 = -10$, $\tilde{a}_1 = 0.75$ and $\tilde{a}_2 = 1.25$ with similar repulsive and attractive strength parameters but different inverse range parameters; (ii) $\tilde{v}_1 = 1$, $\tilde{v}_2 = -50$, $\tilde{a}_1 = 1.75$ and $\tilde{a}_2 = 0.25$ with a large negative value of attractive strength parameter and a small value for its corresponding inverse range parameter.

5.1. Bound states and resonances

The singularities of the transition matrix located on the imaginary axis in the upper half-plane of the complex q-plane are considered the ‘bound states’, while the poles on the negative imaginary q-axis correspond to the ‘virtual states’. In the lower half q-plane the poles with opposite real parts are symmetrically arranged about the imaginary axis, called the ‘resonance’ and ‘antiresonance’ poles of $T^{(\ell)}$, in the fourth and third quadrants, respectively. For each angular momentum quantum number ℓ, the qualitative features of these singularities depend on the values of three parameters $\tilde{v}_1$, $\tilde{v}_2$ and $\tilde{a}_1$ (or $\tilde{a}_2 = 2 - \tilde{a}_1$).

The *T*-matrix constructed in equation (4.5) explicitly shows the contributions from the bound states and resonances in the complex-energy (or momentum) plane. Figure 1 shows the distribution of poles of the partial-wave transition matrix in the complex $\tilde{q}$-plane for p-wave ($\ell = 1$), d-wave ($\ell = 2$), f-wave ($\ell = 3$) and g-wave ($\ell = 4$) scattering. The multiplicity of the zero of Fredholm determinant $D^{(\ell)}$ located on the imaginary axis in the upper half-plane of the complex q-plane is equal to the degeneracy of the bound states. It may therefore be simply said that the number of zeros of $D^{(\ell)}$ in the upper half of the q-plane equals

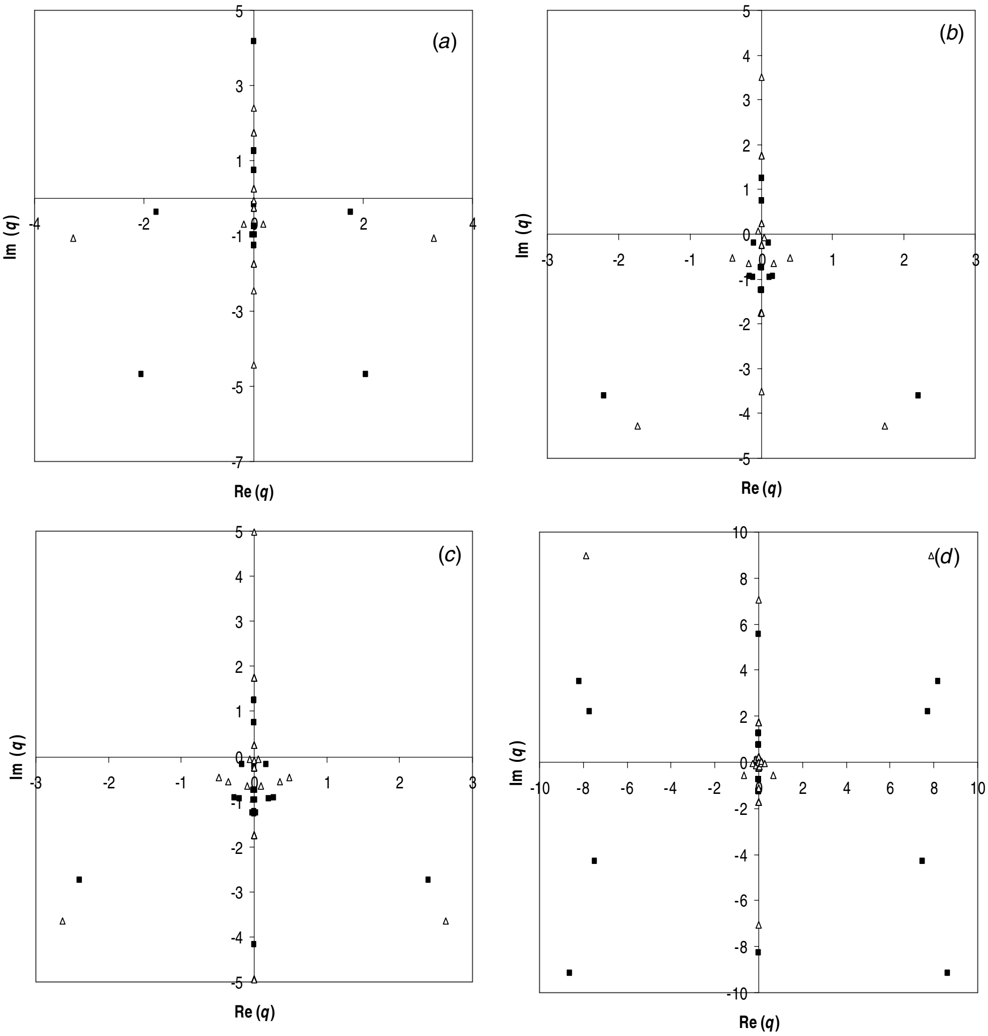

Figure 1. Poles of T-matrix for (a) $\ell = 1$, (b) $\ell = 2$, (c) $\ell = 3$ and (d) $\ell = 4$. The selected values for the potential parameters are: (■) $\tilde{a}_1 = 0.75$, $\tilde{v}_1 = 10$, $\tilde{v}_2 = -10$; (△) $\tilde{a}_1 = 1.75$, $\tilde{v}_1 = 1$, $\tilde{v}_2 = -50$.

the number of bound states, counting each zero as many times as its multiplicity and each bound state as many times as its degeneracy. For this potential model, it can be seen that there are $2(\ell + 1)$ degenerate bound states at both points $\tilde{q}_k = \mathrm{i}\tilde{a}_1$ and $\tilde{q}_k = \mathrm{i}\tilde{a}_2$ with the energies $E_{B,i} = -\frac{\hbar^2}{2m}\left(\tilde{a}_i^2/a_{av}^2\right)$ and in some cases there is only one non-degenerate bound state.

Moreover, there exist several poles of T-matrix in the fourth quadrant located at $\tilde{q}_k = \tilde{q}_r - \mathrm{i}\tilde{q}_i$ $(\tilde{q}_r, \tilde{q}_i > 0)$ with $\tilde{q}_i$ sufficiently small, called resonances, with the energy E_R and the corresponding width Γ_R as

$$E_R = \frac{\hbar^2}{2ma_{av}^2}\left(\tilde{q}_r^2 - \tilde{q}_i^2\right) \tag{5.1}$$

$$\Gamma_R = \frac{2\hbar^2}{m\bar{a}^2}\tilde{q}_r\tilde{q}_i \tag{5.2}$$

where $\tilde{q}_r$ and $\tilde{q}_i$ are the real and imaginary parts of $\tilde{q}$, respectively. Even further, each pole $\tilde{q}_k$ in the fourth quadrant (resonance pole) goes with a twin pole $\tilde{q}_k^*$ in the third quadrant, called antiresonance pole.

5.2. Scattering amplitude, phase shift and time delay

The on-energy-shell partial-wave scattering amplitude $f_\ell(\tilde{p})$ is given by:

$$f_\ell(\tilde{p}) = -(2\pi)^2 \lim_{\varepsilon\to 0^+} \langle \tilde{p}|T^{(\ell)}(E + \mathrm{i}\varepsilon)|\tilde{p}\rangle \tag{5.3}$$

where $\lim_{\varepsilon\to 0^+}$ means that ε is positive and goes to zero. The total scattering amplitude, which contains all the scattering information then becomes

$$f(\tilde{p}) = \sum_\ell (2\ell + 1) f_\ell(\tilde{p}) \mathrm{P}_\ell(\cos\theta). \tag{5.4}$$

As a result, by inserting equation (4.5) into equations (5.3) and (5.4) the scattering amplitude and its partial-wave will be obtained.

The partial-wave phase shift δ_ℓ can be expressed in terms of the partial-wave scattering amplitude as

$$f_\ell(\tilde{p}) = 2\mathrm{i}\, \mathrm{e}^{\mathrm{i}\delta_\ell} \sin\delta_\ell \tag{5.5}$$

or alternatively

$$\delta_\ell = \arctan\left[\frac{-\mathrm{Re}\, f_\ell(\tilde{p})}{\mathrm{Im}\, f_\ell(\tilde{p})}\right] \tag{5.6}$$

where $f_\ell(\tilde{p})$ is given in equation (5.3). Therefore, the partial-wave scattering phase shift associated with our potential model can be obtained analytically. Clearly, since the arctangent is a multivalued function, it is necessary to choose a particular branch which makes δ continuous. Figure 2 represents the partial-wave phase shifts as a function of reduced energy defined as $\tilde{E} \equiv m\big/a_{av}^2$ for p- and d-waves.

Another interesting quantity to characterize quantum scattering processes is the time delay. The formation of a resonance, which occurs as an unstable intermediate state in scattering processes, introduces a time delay between the arrival of the incident wave and its departure from the collision region. This property has been examined in many works [13, 22]. The partial-wave time delay $\tau_\ell \equiv \frac{2}{q}\frac{\mathrm{d}\delta_\ell}{\mathrm{d}q}$ can be used to estimate the resonance energy, since τ_ℓ should rapidly rise as a function of momentum (or energy) and reach its peak near a resonance indicating its location. Figure 3 represents the reduced partial-wave time delay $\tilde{\tau}_\ell$ for some selected cases, where $\tilde{\tau}_\ell = a_{av}^2\tau_\ell = 2\mathrm{d}\delta_\ell/\mathrm{d}\tilde{E}$. Obviously, any significant change in phase shift, as a function of energy, will be seen clearly in the time delay plots. It should be noted that a decrease in the phase shift, with respect to energy, would give rise to a negative time delay. According to the Wigner theorem [35], the increase and decrease in the phase shift should balance each other. In fact, according to Levinson's theorem, a sudden jump in the phase has been compensated by a strong negative slope. Wigner found that the negative partial-wave time delays are restricted by 'causality condition' [36]. A detailed interpretation and discussion of the circumstances giving rise to such an observation can be found in [37]. We have seen in figure 3 that in the energy region close to the resonances, the time delay is large. We found that the negative time delay occurs at higher values of attractive contribution to the inverse range parameter ($\tilde{a}_2$).

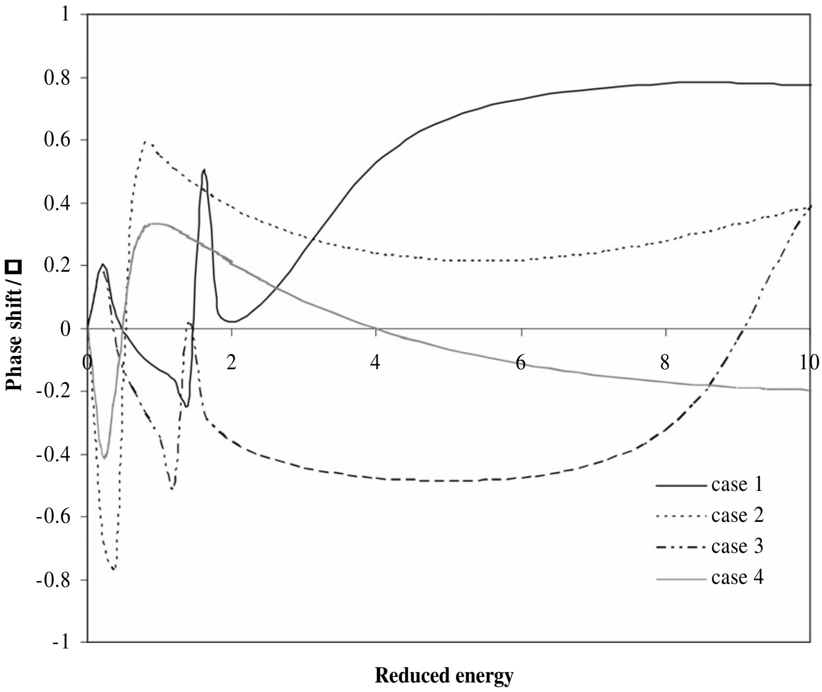

Figure 2. Phase shifts for p-(cases 1 and 2) and d-(cases 3 and 4) waves. Case 1: $\tilde{a}_1 = 0.75$, $\tilde{v}_1 = 10$, $\tilde{v}_2 = -10$; case 2: $\tilde{a}_1 = 1.75$, $\tilde{v}_1 = 1$, $\tilde{v}_2 = -50$; case 3: $\tilde{a}_1 = 0.75$, $\tilde{v}_1 = 10$, $\tilde{v}_2 = -10$; case 4: $\tilde{a}_1 = 1.75$, $\tilde{v}_1 = 1$, $\tilde{v}_2 = -50$.

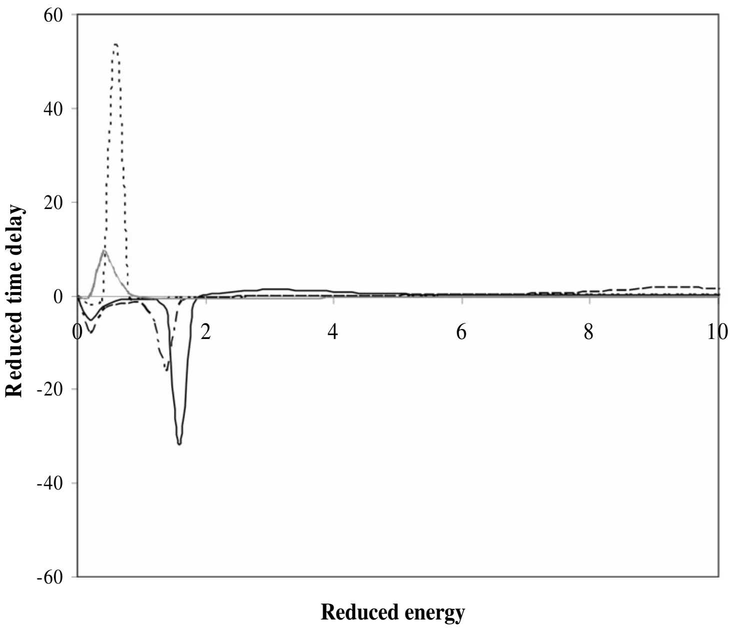

Figure 3. Same as figure 2 for time delays.

6. Conclusions

It is observed that equation (3.10) is an explicit coordinate representation of the ℓth partial-wave scattering wavefunction via the nonlocal rank-two separable potential. The analytic expression of the ℓth partial-wave off-shell transition matrix, equation (4.5), and the explicit expressions for a number of ℓth partial-wave scattering properties, such as scattering amplitude, bound and resonance states, phase shifts and time delays are obtained. The bound and resonance poles, scattering phase shift and time delay for some selected values of angular momentum and inverse range and coupling strength potential parameters are described.

Acknowledgments

Financial support received from the University of Tehran Research Council is greatly appreciated. The authors wish to thank Professor M N Sarblouki and Mr Payam Nasertayoob for helpful comments.

Appendix A

The expressions of $\Lambda^{(\ell)}(\tilde{a}_i, \tilde{a}_j, \tilde{q})$ appearing in equation (3.7) are as follows:

$$\Lambda^{(0)} = 1/2 \tag{A.1}$$

$$\Lambda^{(1)} = (\tilde{q}^2 - \tilde{a}_i\tilde{a}_j) + 2\mathrm{i}(\tilde{a}_i + \tilde{a}_j)\tilde{q} \tag{A.2}$$

$$\Lambda^{(2)} = 3\tilde{q}^4 + 9\mathrm{i}(\tilde{a}_i + \tilde{a}_j)\tilde{q}^3 - 2\left(4\tilde{a}_i^2 + 11\tilde{a}_i\tilde{a}_j + 4\tilde{a}_j^2\right)\tilde{q}^2 - 9\mathrm{i}(\tilde{a}_i + \tilde{a}_j)\tilde{a}_i\tilde{a}_j\tilde{q} + 3\tilde{a}_i^2\tilde{a}_j^2 \tag{A.3}$$

$$\begin{aligned}\Lambda^{(3)} = {} & 10\tilde{q}^6 + 40\mathrm{i}(\tilde{a}_i + \tilde{a}_j)\tilde{q}^5 - 2\left(29\tilde{a}_i^2 + 73\tilde{a}_i\tilde{a}_j + 29\tilde{a}_j^2\right)\tilde{q}^4 - 16\mathrm{i}(\tilde{a}_i + \tilde{a}_j) \\ & \times \left(2\tilde{a}_i^2 + 9\tilde{a}_i\tilde{a}_j + 2\tilde{a}_j^2\right)\tilde{q}^3 + 2\left(29\tilde{a}_i^2 + 73\tilde{a}_i\tilde{a}_j + 29\tilde{a}_j^2\right)\tilde{a}_i\tilde{a}_j\tilde{q}^2 \\ & + 40\mathrm{i}\tilde{a}_i^2\tilde{a}_j^2(\tilde{a}_i + \tilde{a}_j)\tilde{q} - 10\tilde{a}_i^3\tilde{a}_j^3\end{aligned} \tag{A.4}$$

$$\begin{aligned}\Lambda^{(4)} = {} & 35\tilde{q}^8 + 175\mathrm{i}(\tilde{a}_i + \tilde{a}_j)\tilde{q}^7 - 5\left(69\tilde{a}_i^2 + 166\tilde{a}_i\tilde{a}_j + 69\tilde{a}_j^2\right)\tilde{q}^6 - 25\mathrm{i}(\tilde{a}_i + \tilde{a}_j) \\ & \times \left(13\tilde{a}_i^2 + 47\tilde{a}_i\tilde{a}_j + 13\tilde{a}_j^2\right)\tilde{q}^5 + 2\left(64\tilde{a}_i^4 + 601\tilde{a}_i^3\tilde{a}_j + 1179\tilde{a}_i^2\tilde{a}_j^2\right. \\ & \left. + 601\tilde{a}_i\tilde{a}_j^3 + 64\tilde{a}_j^4\right)\tilde{q}^4 + 25\mathrm{i}\tilde{a}_i\tilde{a}_j(\tilde{a}_i + \tilde{a}_j)\left(13\tilde{a}_i^2 + 47\tilde{a}_i\tilde{a}_j + 13\tilde{a}_j^2\right)\tilde{q}^3 \\ & - 5\tilde{a}_i^2\tilde{a}_j^2\left(69\tilde{a}_i^2 + 166\tilde{a}_i\tilde{a}_j + 69\tilde{a}_j^2\right)\tilde{q}^2 - 175\mathrm{i}\tilde{a}_i^3\tilde{a}_j^3(\tilde{a}_i + \tilde{a}_j)\tilde{q} + 35\tilde{a}_i^4\tilde{a}_j^4\end{aligned} \tag{A.5}$$

$$\begin{aligned}\Lambda^{(5)} = {} & 126\tilde{q}^{10} + 756\mathrm{i}(\tilde{a}_i + \tilde{a}_j)\tilde{q}^9 - 14\left(134\tilde{a}_i^2 + 313\tilde{a}_i\tilde{a}_j + 134\tilde{a}_j^2\right)\tilde{q}^8 - 84\mathrm{i}(\tilde{a}_i + \tilde{a}_j) \\ & \times \left(29\tilde{a}_i^2 + 94\tilde{a}_i\tilde{a}_j + 29\tilde{a}_j^2\right)\tilde{q}^7 + 6\left(281\tilde{a}_i^4 + 2062\tilde{a}_i^3\tilde{a}_j + 3772\tilde{a}_i^2\tilde{a}_j^2 + 2062\tilde{a}_i\tilde{a}_j^3\right. \\ & \left. + 281\tilde{a}_j^4\right)\tilde{q}^6 + 8\mathrm{i}(\tilde{a}_i + \tilde{a}_j)\left(64\tilde{a}_i^4 + 865\tilde{a}_i^3\tilde{a}_j + 2169\tilde{a}_i^2\tilde{a}_j^2 + 865\tilde{a}_i\tilde{a}_j^3 + 64\tilde{a}_j^4\right)\tilde{q}^5 \\ & - 6\tilde{a}_i\tilde{a}_j\left(281\tilde{a}_i^4 + 2062\tilde{a}_i^3\tilde{a}_j + 3772\tilde{a}_i^2\tilde{a}_j^2 + 2062\tilde{a}_i\tilde{a}_j^3 + 281\tilde{a}_j^4\right)\tilde{q}^4 \\ & - 84i\tilde{a}_i^2\tilde{a}_j^2(\tilde{a}_i + \tilde{a}_j)\left(29\tilde{a}_i^2 + 94\tilde{a}_i\tilde{a}_j + 29\tilde{a}_j^2\right)\tilde{q}^3 + 14\mathrm{i}\tilde{a}_i^3\tilde{a}_j^3\left(134\tilde{a}_i^2 + 313\tilde{a}_i\tilde{a}_j\right. \\ & \left. + 134\tilde{a}_j^2\right)\tilde{q}^2 + 756\mathrm{i}\tilde{a}_i^4\tilde{a}_j^4(\tilde{a}_i + \tilde{a}_j)\tilde{q} - 126\tilde{a}_i^5\tilde{a}_j^5\end{aligned} \tag{A.6}$$

References

[1] Elberfeld W and Kleber M 1988 *Am. J. Phys.* **56** 54
[2] Snider R F 1988 *J. Chem. Phys.* **88** 6438

[3] Muga J G and Snider R F 1990 *Can. J. Phys.* **68** 403
[4] Muga J G and Palao J P 1998 *J. Phys. A: Math. Gen.* **31** 9519
[5] van Dijk W, Kermode M W and Moszkowski S A 1998 *J. Phys. A: Math. Gen.* **31** 9571
[6] Osman A 1979 *Phys. Rev.* C **19** 1127
[7] Dasgupta S, Mazumdar I and Bhasin V S 1994 *Phys. Rev.* C **50** R550
[8] Ueta K, Miyake H and Bund G W 1999 *Phys. Rev.* C **59** 1806
[9] Yamaguchi Y 1954 *Phys. Rev.* **95** 1628
[10] Yamaguchi Y and Yamaguchi Y 1954 *Phys. Rev.* **95** 1635
[11] Mitra A N 1961 *Phys. Rev.* **123** 1892
[12] Mitra A N and Anand J D 1963 *Phys. Rev.* **130** 2117
[13] Mitra A N and Bhasin V S 1963 *Phys. Rev.* **131** 1265
[14] Tabakin F 1964 *Ann. Phys.* **30** 51
[15] Tabakin F 1968 *Phys. Rev.* **174** 1208
[16] Mongan T R 1966 *Phys. Rev.* **147** 1117
[17] Mongan T R 1968 *Phys. Rev.* **175** 1260
[18] Arndt R A and MacGregor M H 1966 *Phys. Rev.* **141** 873
[19] Ghirardi G C and Rimini A 1964 *J. Math. Phys.* **5** 722
[20] Ali S, Rahman M and Husain D 1974 *Phys. Rev.* C **9** 1657
[21] Talukdar B, Das U and Mukhopadhyaya 1979 *J. Math. Phys.* **20** 372
[22] Daskaloyannis C 1981 *J. Math. Phys.* **22** 1861
[23] Van Haeringen H 1983 *J. Math. Phys.* **24** 1274
[24] Muga J G, Wei G and Snider R F 1996 *Ann. Phys.* **252** 336
[25] Muga J G, Wei G and Snider R F 1996 *Eur. Phys. Lett.* **35** 247
[26] Adhikari S K and Ghosh A 1998 *J. Phys.: Condens. Matter* **10** 135
[27] Abarenkov I V and Tupitsyn I I 2001 *J. Chem. Phys.* **115** 1650
[28] Alhaidari A D 2001 *J. Phys. A: Math. Gen.* **34** 11273
[29] Kidun O, Fomiykh N and Berakdar J 2002 *J. Phys. A: Math. Gen.* **35** 9413
[30] de Prunelé E 1997 *J. Phys. A: Math. Gen.* **30** 7831
[31] Maghari A and Tahmasbi N 2005 *J. Phys. A: Math. Gen.* **38** 4469
[32] Tahmasbi N and Maghari A 2007 *Physica* A **382** 537
[33] Arfken G 1985 *Mathematical Methods for Physicists* (New York: Academic)
[34] Newton R G 1982 *Scattering Theory of Waves and Particles* (New York: Springer)
[35] Wigner E P 1955 *Phys. Rev.* **98** 145
[36] Nussenzveig H M 1972 *Causality and Dispersion Relations* (New York: Academic)
[37] Muga G, Sala Mayato R and Egusquiza I L 2008 *Time in Quantum Mechanics* (Berlin: Springer)